LINEAR ALGEBRA

Week 5

Systems of Homogeneous Linear Equations

Definition 1. A linear system $AX = O$ is called a homogeneous. $X = O$ is called the trivial solution of a homogeneous system $AX = O$.

Existence of nontrivial solutions to $AX = O$ is equivalent to existence of a free variable of the system.

Definition 2. When a nontrivial solution to $AX = O$ exists, we fix each free variable once, assign it 1 and all others free variables 0 in order to get a fundamental solution to $AX = O$.

Thus the number q of fundamental solutions $X_1, \cdots, X_q$ to the homogeneous system $AX = O$ is equal to the number of its free variables $x_{k_1}, x_{k_2}, \cdots, x_{k_q}$ and the general solution to $AX = O$ is:

$$X = x_{k_1} X_1 + x_{k_2} X_2 + \cdots + x_{k_q} X_q .$$

Exercise 1. Solve the homogeneous system: $\begin{cases} x - y + z - t = 0 \\ 2x - 3y + 2z - 4t = 0 \\ 3x - 2y + 3z - 2t = 0 \\ y + t = 0 \end{cases}$.

Solution. Lets transform the coefficient matrix to the echolon form:

$$\begin{matrix} x & y & z & \text{t} \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & -3 & 2 & -4 \\ 3 & -2 & 3 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[(-3)R_1 + R_3]{-2R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3 + R_4]{R_2 + R_3} \begin{bmatrix} \boxed{1} & -1 & 1 & -1 \\ 0 & \boxed{-1} & 0 & -2 \\ 0 & 0 & 0 & \boxed{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the free variable of the system is z. We obtain the fundamental solution X corresponding to $z = 1$:

$$\begin{cases} x-y+1-t=0 \\ y+2t=0 \\ t=0 \end{cases} \Rightarrow x=-1, y=0, t=0$$

So $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the general solution is $X = zX_1 = z\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \\ 0 \end{bmatrix}$.

Exercise 2. Solve the system:

$$\begin{cases} x+y-z+t-u=0 \\ z-t=0 \\ t+u=0 \end{cases}.$$

Solution The coefficient matrix of the system: $\begin{bmatrix} \boxed{1} & 1 & -1 & 1 & -1 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}$

The free variables are y and u. Consturct the corresponding fundemental solutions:

1) $y=1$ and $u=0$. So

$$\begin{cases} x+1-z+t=0 \\ z-t=0 \\ t=0 \end{cases} \Rightarrow t=0, z=0, x=-1, y=1, u=0 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2) $y=0$ and $u=1$

$$\begin{cases} x-z+t-1=0 \\ z-t=0 \\ t+1=0 \end{cases} \Rightarrow t=-1, z=-1, x=1, y=0, u=1 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the general of the system is $X = yX_1 + uX_2 = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

Exercise 3. Find the general solution to the following system in terms of its fundamental solutions:

$$\begin{cases} y - z + u = 0 \\ x - 2z + 3t = 0 \\ y + t = 0 \end{cases}.$$

Solution $\begin{bmatrix} 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[-R_2+R_3]{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 \\ 0 & \boxed{1} & -1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 & -1 \end{bmatrix}$

t and u are free variables. So the fundamental solutions are

1) $t = 1$ and $u = 0$.

$$\begin{cases} x - 2z + 3 = 0 \\ y - z = 0 \\ z + 1 = 0 \end{cases} \Rightarrow x = -5, y = -1, z = -1, t = 1, u = 0 \Rightarrow X_1 = \begin{bmatrix} -5 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

2) $t = 0$ and $u = 1$

$$\begin{cases} x - 2z = 0 \\ y - z + 1 = 0 \Rightarrow x = 2, y = 0, z = 1, t = 0, u = 1 \Rightarrow X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ z - 1 = 0 \end{cases}$$

So the general solution of the system is $X = tX_1 + uX_2 = t\begin{bmatrix} -5 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 4. Find the general solution of $\begin{cases} x + y - z + t - u + v = 0 \\ x - y + z + t + u - v = 0 \end{cases}$.

Solution.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 & -1 & 1 \\ 0 & \boxed{-2} & 2 & 0 & 2 & -2 \end{bmatrix} \xrightarrow{\left(\frac{-1}{2}\right)R_2} \begin{bmatrix} \boxed{1} & 1 & -1 & 1 & -1 & 1 \\ 0 & \boxed{1} & -1 & 0 & -1 & 1 \end{bmatrix}$$

z, t, u and v are free variables. So fundamental solutions are

$$1)\ z=1,\ t=0, u=0, v=0 \Rightarrow \begin{cases} x+y-1=0 \\ y-1=0 \end{cases} \Rightarrow x=0, y=1 \Rightarrow X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2)\ z=0,\ t=1, u=0, v=0 \Rightarrow \begin{cases} x+y+1=0 \\ y=0 \end{cases} \Rightarrow x=-1, y=0 \Rightarrow X_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$3)\ z=0,\ t=0, u=1, v=0 \Rightarrow \begin{cases} x+y-1=0 \\ y-1=0 \end{cases} \Rightarrow x=0, y=1 \Rightarrow X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$4)\ z=0,\ t=0, u=0, v=1 \Rightarrow \begin{cases} x+y+1=0 \\ y+1=0 \end{cases} \Rightarrow x=0, y=-1 \Rightarrow X_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the general solution of the system is

$$X = zX_1 + tX_2 + uX_3 + vX_4 z \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Exercise 5. What must be value of k if the system has two fundamental solutions. Find them.

$$\begin{cases} x+ky-2z+(k+2)t=0 \\ z+3t=0 \\ z+(k+1)t=0 \end{cases}.$$

Solution. $\begin{bmatrix} 1 & k & -2 & k+2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & k+1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} \boxed{1} & k & -2 & k+2 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & \boxed{k-2} \end{bmatrix}$

If we have two fundamental solutions, then we have two free variables. Hence if we take $k=2$, y and t are free variables and the coefficient matrix should be

$$\begin{bmatrix} 1 & 2 & -2 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1) $y=1, t=0$

$$\left.\begin{matrix} x+2-2z=0 \\ z=0 \end{matrix}\right\} \Rightarrow z=0, x=-2 \Rightarrow X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2) $y=0, t=1$

$$\left.\begin{matrix} x-2z+4=0 \\ z+3=0 \end{matrix}\right\} \Rightarrow z=-3, x=-10 \Rightarrow X_2 = \begin{bmatrix} -10 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

(The general solution of the system is $X = yX_1 + tX_2 = y\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -10 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

But this question is not asking for it.)

Let $AX = B$ be a linear system. When it is consistent, its general solution can be written in the form $X = X_g + X_p$, where X_g is the general solution to the homogeneous system $AX = 0$ and X_p is a particular solution ($=$ any solution) to $AX = B$.

Example 6. Find the general solution to the following system (in the case if it is consistent).

$$\begin{cases} x + 2y + 3z = 1 \\ x + z = 1 \end{cases}.$$

Solution. $\left[A|B\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-2} & -2 & 0 \end{array}\right] \xrightarrow{\left(-\frac{1}{2}\right)R_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right]$

$z = 1$

$$\begin{cases} x + 2y + 3 = 0 \\ y + 1 = 0 \end{cases} \Rightarrow x = -1, y = -1 \Rightarrow X_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$X_g = zX_1 = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The particular solution of the $AX = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is, for example, $z = 0 : X_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$X = X_g + X_p = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - z \\ -z \\ z \end{bmatrix}.$$

Exercise 7. Find the values of k for which the matrix equation

$$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

has a solution, and find, for these k, the general solution to the given equation.